# THE YEAR'S PROGRESS IN MATHEMATICS IN THE UNIVERSITY HIGH SCHOOL

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The number of capable teachers of secondary mathematics, who are striving to improve the educational quality of secondary-school work is already large and it is rapidly growing. Good secondary teachers are coming to regard it a professional duty to try each year one or more definite things that promise to improve the teaching, to benefit the taught, or to enrich the quality of the subject-matter of their specialty. Attitudes of mind have changed during recent years, and to advocate progress in mathematical teaching has grown half-popular. Enlightened opinion today recognizes—nor fears the academic ban—both the need and practicability of marked improvement in the matter and method of secondary-school mathematics.

The recent awakening of pedagogic interest in mathematics is already accomplishing much in the textbook line. More than a score of secondary texts of the subject, all bearing the imprint of a new order, and some of them from publishing houses that are accustomed to cloaking *commercialism* under the ermine of *conservatism,* and all in a brief twelvemonth, is highly encouraging. The pools of tradition are being stirred to their depths. The official representative of a house that is nothing if not conservative said in substance to the writer the other day: "There are getting to be so many who are wanting the mixed type of mathematics in the high school, that we had to get a book written to cater to this audience." The easy-going unconcern for innovation that afflicted teachers of mathematics only a brief time since now appears in print only rarely and in places so out-of-the-way as to work little harm.

And what has been achieved is but an earnest of greater and better things in the next decade. Traditional rigidity, institu-

tional conservatism and pedantic dread of change are ceasing to be obstacles. Modern life and its needs, the modern boy and his needs, are year by year being less frowned upon in the schoolroom. The self-constituted flag-men along the route of progress are less hysterical and more sane in signaling the dangers of rapid change in school mathematics. Many of them are even getting on board. In numerous ways professional life to progressive teachers of mathematics is becoming more and more worth while. For all these blessings let us be duly grateful!

The High School of The University of Chicago lays no claim to exclusive consideration as an experimental laboratory for dealing with practical problems of secondary mathematical education. It does, however, claim to be such an institution as this and it is doubtful whether any such laboratory enjoys freer and more favorable conditions for its legitimate work of experimentation than exist here. One of the ambitions common to its teachers and administrative officers is that this peculiarly favorable environment shall redound in the fullest possible measure and in the largest way to the most substantial good of the public secondary education of the nation. The determination of those responsible for the mathematical instruction is to leave no part of their peculiar duty undone that may serve to promote the gratification of the common ambition to aid every legitimate advance. Under such circumstances avowals of faith, and citations of progress made from time to time on the part of the mathematical corps cannot fail to attract some professional interest.

The mathematical corps of the University High School keeps two distinct purposes before it, viz.: (1) By the aid of its peculiarly favorable equipment to work out practically and to organize into generally practicable texts the best possible type of subject-matter for secondary students of mathematics; (2) To make its classrooms function both as laboratories and as exhibit rooms for the study of the best possible type of secondary teaching both for actual and for intending teachers of secondary mathematics.

In its attempt to meet this double purpose the efforts of the

mathematical corps are ever concentrated upon what is *practical* and *practicable* under conditions prevailing in the public secondary schools in general. From certain remarks pertaining to some statements the writer made in an article in the issue of the *Review* for October, 1907, it would seem that some look upon the purposes and problems of the University High School as having only a narrowly specialized interest. Let it therefore be stated plainly that the mathematical corps is seeking nothing more narrow than the best type of mathematical education for the public secondary schools of our country. There is no special feature in our situation to restrict in any way the significance of our findings. The students of the High School come mainly from the public schools; from the most varied class of homes; they represent the most widely varied conditions of society, and are of a highly cosmopolitan character. Many are preparing for college and many more are preparing for non-collegiate pursuits. The purpose of the school is to give its students the best type of education for the period of life from fourteen to eighteen years of age.

Perhaps the only restrictive element in our situation that is peculiar is that a larger percentage of our students than is common in high schools generally are preparing definitely for colleges. But this is the limitation of high-school autonomy about which high-school men complain most loudly, and if it rests more heavily upon us than upon others, our educational findings ought to have even greater practicability and workability elsewhere than here. We do not feel, however, that this circumstance is so objectionable as many hold it to be. All this is but another reason why what is achieved here should be of general interest.

In the report made in the *School Review* in October, 1907, under the title, "The Year's Progress," it was stated that the first three or four weeks at the beginning of the first year would be set aside hereafter for the explicit purpose of reviewing the arithmetic topics in which beginners are always found deficient. In addition to this review, an attempt was to be made to accomplish something of a scientific filling out of the topics of arith-

metic with a view to bringing them under better working control.

After a year's experience in this matter it seems much better to accomplish this review under the guise of algebra. Beginners hold themselves somewhat in abeyance at the thought of going over again what they have grown tired of. They feel that they have done their arithmetic work satisfactorily, to their former teachers at least, and that to call upon them to go over this work again is to discredit the preparation upon which they have been admitted to the class. This feeling prejudices appreciably the point of view of the beginner toward his high-school work. It seems better to escape it by a type of algebraic work which, while perhaps it looks like advanced work, and in a sense is advanced work, is at the same time a *re*-view of the arithmetic. This makes the beginning work, which is difficult enough at best, much smoother, more palatable, and, on the whole, more encouraging at the outset than does a frank going over of the grade-work in arithmetic. Accordingly, exercises for testing and reviewing the addition and multiplication facts of arithmetic are given in the form of equations for mental treatment to find $x$, like the following:

$$\begin{aligned} 2x+x&=9 \\ 7x+6&=62 \\ 8x-4&=52 \\ 11x-2x+3x&=108, \text{ etc.} \end{aligned}$$

For review of the laws of arithmetical calculation are given numerous oral exercises to find the value of the letter, such as these:

$$\begin{aligned} 4x-3&=12\ ; \\ 16t+2t-13t&=22\tfrac{1}{2}\ ; \\ 6s-3.5s+5.5s&=68\ ; \\ \frac{x}{2}+\frac{x}{4}&=3\ ; \\ \frac{t}{3}-\frac{t}{6}&=10\ ; \\ \frac{15}{a}&=5\ ;\quad \text{etc.} \end{aligned}$$

Such exercises are given continually and systematically for several days, until the arithmetical processes are well mastered.

Incidentally, the use and office of the equation is being *sensed.*

It was also stated in the report of progress of October, 1907, that the plan of paralleling the geometry of the second year with a course in algebra was unsatisfactory, (1) because of the pupils' dislike for the plan of being led from one subject to the other at what seemed to them the whim of the teacher; (2) because the teachers themselves felt that they could not make their case with the pupils clear enough to justify this abrupt change, and (3) because American textbooks are poorly planned for parallel work. On the whole, under the circumstances then existing, the results of the year's attempt to carry on parallel courses in algebra and geometry in the second year were reported not favorable to the plan.

From the remarks that have come to him concerning the report, the writer feels that he overstated the case to the disadvantage of the parallel plan. Not one of the teachers who tried the plan felt that the trial was a fair one. This was conceded freely and immediately by everyone who tried the plan. For the reason that the plan was imperfectly worked out beforehand, and because of the general difficulties of presenting the work to advantage with the materials and texts that had to be used, more time was felt by all to be necessary before a satisfactory, and anything like a final judgment could be formed.

Pupils of the second year are still little more than children. They are highly dependent in their modes of thinking upon teacher, book, and external help. The parallel plan under existing conditions, to be successful, requires at least a modicum of independence and maturity. It has been determined therefore to test the parallel plan more fully and more fairly by applying it this year in the third-year classes of the University High School. We are, therefore, giving the class time of this year in about equal portions to trigonometry and advanced algebra. The teachers are reporting results to be highly satisactory. They feel that because the students here are more independent in their thinking, because they are already in possession of the outlines of the sciences of algebra and geometry, and perhaps because the teachers themselves, who have now had considerable experience

in ways of putting into correlation the facts of algebra and geometry, are more skilful than they were two years ago, and because of the innate virtues of the plan of keeping the twofold point of view always present, the parallel plan promises a high degree of success in the third-year work.

Encouraged by this promised success in the third year, our teachers are rapidly coming to feel that the reasons for dissatisfaction in the second year, as the plan was attempted, may be easily obviated. The two lines of work must be more closely associated and interconnections should be more definitely and more strongly made for the second-year pupils. By giving more attention to the subject-matter with reference to accomplishing these ends, the parallel plan, with numerous interconnecting passages back and forth between the two fields of algebra and geometry, will much improve results in the second year. A body of subject-matter is already begun by the high-school teachers with the purpose of facilitating the parallel plan here. With even measurably suitable texts the parallel plan will assuredly succeed admirably also in the second year.

The first year's work has been based essentially upon the little book, *First-Year Mathematics* (The University of Chicago Press), but many supplementary developments and adjustments which deviate from the little book of notes have improved things materially. The revision of the notes—for the book was little more than a body of notes—into what will be a *Textbook of First-Year Mathematics,* is now far along and is working admirably in the High School classes.

It may be well to specify a few of the alterations that experience has shown to be desirable. The fundamental thought is that in the first year's work the high-school pupil shall face his problems as mathematical problems, rather than as algebraic or geometric problems as such. The specialized point of view is too highly sophisticated for beginners. It is clear to all that much more geometry, both as to form and subject-matter, is needed in the first year, and the text which is to appear soon from the University Press will emphasize the geometric point of view and mode of attack much more fully than does the little

book now in use. The body of mathematical material making up the work of the first year is something more than mere mixed mathematics. In numerous ways the material has been correlated into a more unified than merely a mixed body of truth, though even a mixed type of subject-matter keeps the twofold point of view of both algebra and geometry continually in mind.

It seems clear that the geometric mode of procedure impresses the beginner more deeply than does the algebraic. Reasons seem to rest more firmly upon substantial bases of concepts in geometric work than in the customary isolated type of algebraic work. The pupil readily carries the faith in the soundness and reasonableness of mathematical processes gained in geometric work into the algebraic domain. He comes to feel that whether he sees the *rationale* of the process of algebraic treatment or not, there *is* a *rationale,* and that he ought to get hold of it. His attitude toward his work has in it at least a tinge of morality. He cares more to understand the algebra. He confronts the algebraic work—which is made the organizing line of ideas through the first year—more seriously, more interestedly, and more profitably than he does by devoting exclusive attention to wholly algebraic beginnings.

Another improvement, which the study of high-school work has shown to be desirable in the interest of education, is making itself felt for good in the high school. Few high-school teachers have failed to feel that it is distinctly unwise to condense all the mathematical work of the high school into the first two years, as is customary, and to allow mathematical interests, as such, to be taken care of for the next two years, either not at all, or indirectly through certain subjects calling here and there, in a random way, for the application of a little mathematics. The last two years of high school lie as a long gap between the mathematical work which the pupil has at best but poorly mastered during the first two years—the crudest period of the high school—and the time when he must face the severe tests put upon him by college-entrance examinations, or by the demands of industrial and commercial life. In short, as the work is customarily programmed, the two most immature years of

the high school are given to this most highly matured branch of study of the whole curriculum. Then come two years of doing little or nothing in mathematics, and then come the tests. It is little wonder that the high-school pupil often disappoints his high-school teacher and administrative officers by the way he fails to meet the tests that are put upon his work. The wonder is rather that he meets them as well as he does.

As a step in the right direction of removing this difficulty, this year but four hours a week are being given to mathematics in the first year, and every student of the fourth year is being required to take one hour per week of work that will call up and renew his former hold on mathematics, and perhaps strengthen it a little. It is thought this plan may aid the high-school graduate to know what he knows when he needs it.

The method of procedure followed in working out the foregoing plans in the classroom is now and has been to develop the outlines of the theory of the several subjects with—not for—the class, and to assign supplementary work for the study-hour or for home-study that is of the nature of fuller developments of the theory, or of applications of the theory, or of exercises to deepen and to impress this theory. If by laboratory plan, work of an individual character in the classroom under the supervision of an expert may be understood, then we may successfully claim to be making considerable use of it. But with this plan as with all devices for accomplishing the ends of mathematical education, we use it but do not permit it to dominate instruction. When a little lecturing, or a little free-and-easy chatting over difficulties, or a little exhibiting or exemplifying of the way to resolve difficulties seems to be the most economical procedure, no teacher hesitates to resort to it. The heuristic method, the laboratory method, the socratic method, the inductive method, the deductive method, and numerous others are in the best of repute with us, but, of course, they are regarded as mere tools to be used as occasion shows them to be economical or otherwise desirable. No one method, nor even many of them taken together, are looked upon as of sufficient educational value to warrant their exclusive jurisdiction over the work of instruction.

We insist upon the open-door policy as to mathematical method. One teacher reports that the greatest benefit he has derived during the year is the firmness with which he has become convinced that extensive use of the inductive plan is best for beginners in secondary mathematics. Another feels that it is his most important finding that a combination of the inductive and deductive methods is best, as the transition from induction to deduction, which must be made, can best be effected by laying hold freely of whatever type of treatment seems most perspicuous to the learner with the particular subject under consideration. Another finds that the nature of the subject, or problem, largely determines its method.

In addition to the effects on the curriculum, the effects in the way of a better attitude on the part of the pupils toward the study of mathematics have been highly gratifying. Most of the early criticisms that pupils made to the unified, or mixed type of mathematics were intended as mere compliments to the views they knew to be held by the grown-ups, to whom the criticisms were made. In the writer's opinion, and his view is shared by many others, the mathematical spirit among the students is more friendly and more positive now than it has been for years. May the nature of the subject-matter not claim a little credit for this happy state of affairs?

But I must not omit to specify the effect which the past year's work has had upon the teachers themselves. It seems best to allow these teachers to speak each in his own way. The writer has requested each one to specify as he chooses what the year has wrought for him. The reader of this report may, therefore, judge for himself as to whether or not the more spirited work of teachers who are holding themselves in the experimental frame of mind is worth while. Many have remarked that in the public schools experiments cannot be attempted because time is too expensive for experiments, which might perhaps lead to negative results. I am sure this view is unwarranted and will yet be seen to be so by public-school men. The spirit, the alertness, the *verve* begotten by a half-dozen—or by even two—capable teachers working in concert, perhaps with a touch of the

spirit of rivalry, will produce results so rich and so copious in character, that the time lost in experiments which issue now and then negatively will be many times atoned for, both in the quality and in the quantity of the results. The number of pedagogical experiments that issue in a negative outcome may be reduced almost to the vanishing point by carefully maturing them beforehand. It seems to be the blessing of practical pedagogics that an experimenter who believes in a plan and tries to make it work will certainly succeed. Ninety points of success out of a hundred are faith, works, and will. The experiment may prove to be a success only with the individual teacher, but this safeguards the interests of the pupils upon whom the experiment is tried.

I give below, without signatures, and without comment, the responses made by the University High School teachers to the request that they state what the year has done for them as teachers of secondary mathematics:

One high-school teacher writes:

Some experimenting has been done in the teaching of plane geometry the past year with the object of starting the course more satisfactorily than with the sequence of definitions and propositions of standard texts. The difficulty which most students have in beginning formal geometry has led many experienced teachers to believe that the usual course should be introduced by a more or less extended course in constructive and experimental geometry. Some English texts are written on this plan, but in America the standard texts attempt a strictly logical order and treatment from the beginning. The books on experimental geometry that we have are designed primarily for the last years of the elementary school, and are either too elementary for the second year of secondary schools, or they require more time, if used as a prologue to the regular course, than can be spared for them as curricula now stand.

Furthermore, to divide the course in geometry into a first part, which is wholly experimental, and a second part which is wholly theoretical, is doubtful procedure. The opinion is growing that a more satisfactory plan is to start with constructive and experimental work, with the early introduction of some easy demonstrations of a purely theoretical character, using the experimental method in some cases as the best means, for the time, of establishing a theorem, in other cases as an approach to a theorem proved at once theoretically. Moreover, many portions of the kind of experimental geometry needed to preclude formal geometry are so easy and they adapt

themselves so readily to algebraic work based on it, that much of this work should be done in connection with the first-year work in algebra. This experimental geometry in the first year answers the purpose of an introduction to second-year geometry, and serves to enrich the first-year course in mathematics.

A double distribution of this sort between the first and second years of this preliminary, concept-forming work in geometry is a part of the plan now under way at the University High School. While the material is being prepared for this reorganization of the courses in algebra and geometry, the effort has also been made to inject into the usual course in plane geometry, with a standard text in the hands of the class, considerable experimental geometry as well as many algebraic problems based on geometrical theorems. The following plan, a part of which was tried with satisfactory results by the writer, is offered as a suggestion of how this may be carried out without doing violence to the unity of the course as found in the ordinary textbook. The numbers refer to the sections in Sanders' Plane Geometry:

I. Theorems assumed: 40, 41, 49, 92, 108, 111, 177, 181, 182, 189.
II. Theorems treated experimentally only: 30, 35, 60, 73, 115, 121, 123, 127, 128.
III. Theorems treated both experimentally and theoretically: 47, 53, 55, 77, 81, 85, 89, 94, 100, 138, 157, 164, 192.

The experimental treatment appropriate to the theorems in II and III, above, is such as is found in Baker, *Elementary Plane Geometry;* Smith, *School Geometry;* Warren, *Experimental and Theoretical Geometry;* Wentworth and Hill, *First Steps in Geometry;* or Campbell, *Observational Geometry.*

In two geometry classes the course was started with the algebraic problems based on geometric material, which are found in *Geometric Exercises for Algebraic Solution* (Chicago: The University Press). By this means considerable first-year algebra was reviewed while the geometric notions and facts were being introduced. It would be easy and profitable to carry this initial work in *Geometric Exercises* through the first twenty-three pages, introducing experimental proofs for the theorems on which the algebraic problems are based.

As a further step toward bringing the algebra and geometry together the problems in *Geometric Exercises* were used throughout the year in all geometry classes with satisfactory results.

The past year's experience in first-year classes has shown that experimental geometry in the first year is desirable from the point of view of first-year algebra as well as from that of second-year geometry. The geometry adds concreteness and variety to the first-year course, and arouses the interest of students, methods of presentation being partly experimental and partly theoretical. Experience in teaching the small amount of geometrical matter at present provided for in *First-Year Mathematics* has

convinced the instructors that it is not only practicable but highly desirable to put more in the first year. In some of the classes it was found profitable to use, in addition to the geometrical matter of the text, "algebra-geometry" problems from the first twenty-three pages of *Geometric Exercises,* the instructor presenting experimental evidence for the truth of theorems needed. In the revision of *First-Year Mathematics* it is accordingly planned to add considerable geometrical work, connecting with it numerous algebraic problems based on the geometry.

An argument in favor of a course in geometry for secondary schools in which the theoretical method is introduced, and to some extent accompanied, by the experimental method, is that it has regard for the natural order of procedure of the learning mind. The geometry which has been and is now taught in the secondary schools of this country represents the mature thought of the ancient Greek mathematicians who philosophized on the properties of space. But we are not warranted in assuming that the youth of fifteen or sixteen years of age has the interest or point of view of the mature philosopher, just because he begins the study of geometry on a certain day. Quite a different view is more nearly correct. It would be better to assume that just as the mature reflection of the Greek mathematicians was preceded by the empirical thinking of the Egyptians and Babylonians, so the modern young student, and for even stronger reasons, should begin the study of geometry in an experimental and intuitive way, developing gradually his powers of logical reasoning.

To use both the experimental and theoretical methods in the teaching of geometry, and to distribute the subject-matter of algebra and geometry, with their applications, throughout the first and second years, so as to meet most naturally the needs of the student's growing powers and his interests, is a part of what is being attempted by the department of mathematics of the University High School in its function as an experimental laboratory of the School of Education. The experiments of the past year which are here described have shown that such a reorganization of subject-matter can be made with profit, and that the change from the old to the new can be made without doing violence to the existing order of things.

Another says:

THE WORK OF THE FIRST YEAR

The first four weeks of the first year were devoted to a review of the most fundamental topics of arithmetic. An outline, prepared by a committee of the mathematical department of the High School, was made the basis of the work. After this review, tests in arithmetic were given to all high-school students taking first-year mathematics, and from those who failed an afternoon class in arithmetic was formed. At the same time all were allowed to remain in their regular classes. No homework was required for the special review class, all the work being done under the

supervision of the instructor, assisted by several practice teachers. Students appreciated greatly the opportunity to review a subject in which they knew that they were deficient and several students who *passed* in the test, asked permission to enter this special class and were admitted.

It was found that all of the students of the class were able to do the work, that they improved in the regular classwork and that most of them passed at the end of the quarter. Similar classes are to be formed in the future.

One thing a good teacher must keep in mind is, that he must make the study of mathematics seem interesting and useful to as many of his pupils as possible. He cannot afford to be satisfied with the interest of the few to whom the subject itself may appeal. By bringing in many practical illustrations and by reducing the amount of memory work, I have been more successful this year than in previous years in getting a large number of students interested. It seems that one reason why so many students in upper classes appear to have forgotten so much of the preparatory work is that they memorized rules without having a clear insight into the meaning of them. The following will illustrate: A student in an upper class has difficulty in deciding whether $-2+(-3)$ is equal to $-5$ or $+5$. He dimly remembers a rule "Like signs give plus, unlike signs give minus," but the purport and propriety of the rule are not clear to him. If, instead of merely learning the rule in the first year, he had been taught to think out the result by means of a practical illustration, as: A loss of two dollars followed by a loss of three dollars is a loss of five dollars, until he could get his results just as quickly and easily as if he were working with a rule, he would be in less danger of becoming confused in deciding for himself whether $-2$ plus $-3$ is equal to $-5$ or to $+5$. Less emphasis should be placed upon the rules, but more upon thinking out the underlying reason. This plan seems to me to bring better results and to make the work less formal and much more enjoyable to the pupils.

The textbook, *First-Year Mathematics*, with all its weaknesses, proved to be a great help. It relates the subject-matter to other sources of education and to matters of everyday life, thus arousing an interest which otherwise would be difficult to obtain. New topics are approached inductively and concretely. By emphasizing matters of conception, something is left to the imagination and relief is brought to the mind. Thus the student does stronger work than he could if concrete illustration were not given.

The symbolic expression for a law is usually made about the last step in the treatment of a subject. It grows out of the verbal statement, and the symbols are suggested largely by the students themselves, as a means of getting the statement of a law that is quickly understood, and easily referred to for subsequent needs.

By changing the order of many topics, some of the objections against the test were removed.

THE WORK OF THE SECOND YEAR

The work of the second year was introduced by a view of the practical side of the subject of geometry. The aim of the course was made clear. A knowledge of many geometrical concepts was obtained from the study of solids, known to the students, as the cube, the tetraedron, the cone, the cylinder, the prism, and so forth. Many related or suggested theorems of plane geometry were taught and, both by repeated application and by means of problems leading to algebraic solutions, the pupil learned practically all the theorems of the first book of Euclid. Other theorems were developed by constructing figures and by measuring relations between angles or lines with protractor and ruler. These theorems were then used in solving the problems of the first twenty-five pages in *Geometric Exercises for Algebraic Solution.*

The work of getting the equations and deducing other results from them gradually assumed a demonstrative character, and then the study of the demonstrative proofs was taken up. The study of algebra was kept up throughout the year by solving geometric problems algebraically.

The plan, as outlined above, makes the transition from the first year to the second easy and natural. The student knows from the beginning that algebra and geometry are not to be separated, and that he will be held responsible for what he has learned in both subjects. He must be active from the first day on. He produces, he invents new truths, and what he gets is his own and stays with him. This gives him a feeling of enjoyment, which is hard to obtain by dragging him through a number of theorems, where he is always forced to admit, to repeat, to memorize.

Many students to whom the subject of geometry does not appeal, will become interested when it is presented both from the geometric and algebraic point of view.

Teachers who visited this class frequently asked whether I expected to cover all the work given in the geometry text, besides doing so much of the algebraic work. I think that this class did at least as much geometry as any of my classes in preceding years. Most of the original exercises in Sanders' text and also all problems in the first half of and many of the last part of *Geometric Exercises* have been worked by pupils and explained in the classroom.

One of the features of this class was, that original work was done by some students and results were given to the class; and that lectures and talks on topics which involve good geometry, though not given in the ordinary texts, were given by the instructor and outsiders, who were invited to speak. Such talks arouse much interest and develop an attitude towards geometry such as a pupil has toward a good story, which he wishes to remember for the purpose of telling it again. When this attitude is obtained, much overexplaining and mechanical review work can be omitted.

The work of the second year in the next school year will differ in some respects from that of the past year, as many new suggestions have come up, which could not be tried this time.

### THE WORK OF THE FOURTH YEAR. SOLID GEOMETRY

The aim of the course was made clear at the beginning. Models of all solids to be studied were examined and a way worked out by which the areas and volumes may be obtained. Many of these results, e. g., the lateral area of a prism, were found before taking up the first book in solid geometry. The propositions of this book were proved when they were needed. As soon as a new formula was derived, it was applied to some solid.

One of the features of the course was the construction of solids by the students. Diedral angles, triedral angles, opposite and symmetrical triedral angles, polyedral angles, and so forth, were made of paper or string and used in the recitation together with the picture drawn on the blackboard.

Before drawing on the board figures containing relations of lines and planes in space, models of these figures were made by means of a few sticks and some cardboards. In the beginning every proposition was pictured this way.

The plan of letting the development of solid geometry proceed from the needs of mensuration of solids, and of treating the customary propositions as need for them arises in discovering ways of finding surface areas and volumes of the standard figures, begets a lively interest and will be followed more fully in future classes.

### COLLEGE ALGEBRA

The subject was gathered around two topics: (1) the solution of the equation; (2) the series. By uniting the different topics of college algebra in this manner, much time was saved and, although the class was not strong, more ground was covered than in previous years. A great deal of graphical work was done to interpret the meaning of algebraic results and to assist in the solution of equations. This plan of unifying the subjects of college algebra is commended to teachers of high-school classes.

A third writes:

The new education lies along the paths of manual training and laboratory methods. The hand leads, the perceptive and reasoning faculties follow. The concrete comes before the abstract.

In trying to realize these ideals in teaching geometry in the University High School I have proceeded as follows: Compasses, straight-edge, and paper, unruled, cross-ruled, and for tracing, are in the hands of each pupil at every recitation. In the classroom the pupil is introduced to the geometrical magnitudes, the surface, the line, and the point, by being required to crease paper for the purpose of showing *surface,* and *line* as the inter-

section of surfaces; to draw or crease two intersecting lines, for the purpose of showing and locating *point;* to draw many lines through *one* point; to try to draw more than one straight line through two given points.

The *doing* of these and similar things prepares the pupil to understand the introductory definitions, postulates, and axioms.

In preparation for demonstration of theorems, the figures are drawn by each pupil in class according to the hypothesis (protractors being used at first); then tracing paper is used to show congruency.

The tracing paper, where possible, is made to serve in realizing a truth and comprehending a demonstration. For example: before proving the first proposition in our text, viz., the congruency of triangles having an angle and the including sides respectively equal, the pupils are asked to draw a triangle, then on tracing paper to trace two sides exactly as they lie. The *completion* of the triangle consists in connecting the points that are the ends of the two sides traced, and they *see* that this third side also *must* coincide with the third side of the original triangle.

The pupil is taught early in the course to use the straight-edge and compasses to bisect line and angle, to draw a perpendicular to a line through any given point, and to construct an angle equal to a given angle. When drawing figures in class, either on paper or on the blackboard, the construction with instruments is insisted on, and the principles on which it is based are frequently asked for.

To realize the truth of the theorems on proportionality of sides and the facts about areas, the diagrams are constructed by the pupil on cross-ruled paper, and the units are counted. When the class starts with the same length of line, considerable interest is aroused by a comparison of results.

New propositions are developed by the instructor with the pupils before assignment for study, especially with a view to drawing out from the pupils the starting-points and possible methods of reaching conclusions.

An important element of our geometry work is the assignment of *algebraic problems* based on the propositions demonstrated. The mathematical faculty of the school have all contributed to a book of problems in an effort to secure a series of problems to illustrate all the important theorems. About one-fifth of the time is devoted to this algebraic work. In this way the *meaning* of the *facts* stated in the theorems is made clear, and by repeated use in solving problems the facts are fixed in the memory. Besides, the continuity of algebra from the first year is secured, and the ability to *use* algebra as a tool in other sciences is developed.

Sometimes the whole lesson is given up to the algebraic problems, but usually the algebra and the geometry are portions of the same lesson. From these statements it may be seen that the aim of the instructor is to make *handwork* precede the proposition, and to make it contribute constantly to clear thinking in the abstractions of geometry. The difficulties

lie chiefly in the neglect of pupils to bring to class the materials and instruments needed for the day.

In first-year algebra my experiment has been to develop the value of *testing*, or *checking every process* and *problem.*

The first result to the pupil noted, is that he sees that he is dealing with actual numbers because in the fundamental processes with literal integers and fractions as he substitutes an assumed numerical value for each letter in the general numbers given, and in his result, he *sees* that they represent the same *number.*

Another result is that the pupil obtains a clearer *notion* of the fundamental *processes.* For example, if the problem is to subtract $a^2-2ab+b^2$, from $a^2+2ab+b^2$, he obtains $4ab$, this *means* that $4ab$ is the number which he must *add* to $a^2-2ab+b^2$ to produce $a^2+2ab+b^2$. This *adding* of the remainder to the subtrahend is also done after substituting some assumed definite number for each letter of the general numbers. In a problem in division such as $a^3-8b^3 \div a-2b$, he is required to multiply his result into $a-2b$ to see whether it produces $a^3-8b^3$. By repeatedly performing the multiplications, the fact is impressed on his mind that the quotient is the *number* by which the divisor is to be *multiplied* to produce the dividend, and thus seeing *what* he is to find he is less likely to make a blunder in the process of finding it.

In division of fractions this same fact appears again, and considerable interest has been awakened in seeing the fractional dividend appear out of the complicated forms of the combined divisor and quotient.

Again, the *algebraic notation,* the difference between coefficient and exponent—between $3x$ and $x^3$ and other forms is fixed by repeated substitutions of arithmetic numbers for the letters in testing each result. The algebraic language is thus more rapidly learned and understood.

Again, the meaning of the *solution* of an *equation* or system of equations is made clear in literal equations, for by testing his results he *sees* that his "value of the unknown" put in the place of the unknown letter makes both sides of the equation absolutely the same, e. g., solving for $x$ the equation $\frac{mx}{n}+\frac{nx}{m}=m^2+n^2$, he obtains as a result $x=mn$, substituting to test this result, he gets $\frac{m^2n}{n}+\frac{mn^2}{m}=m^2+n^2$. Reducing the first member he gets $m^2+n^2=m^2+n^2$, plainly an identity. To test the result in the same problem by substituting arithmetic values, let $m=2$ and $n=3$: then as $x=mn$, $x=6$: Substituting these values in the original equation $\frac{2\times 6}{3}+\frac{3\times 6}{2}=4+9$. By reducing the fractional forms we get $4+9=4+9$. Thus the pupil *sees* that the solution of the equation means that giving $x$ the value $mn$ produces a *visible* equality or identity, that the sides of the equation are not only *said* to be equal, but that they are plainly seen by him *to be equal numbers.*

The lively interest shown by first-year pupils in this *constant* testing of results, has been a pleasant *surprise* to me, although I have taught first-year algebra for many years. In the study of *factoring,* the plan of testing *every* result by one or two methods was adhered to in this year's experiment. The pupil, in example after example, *multiplies* his "factors" together to see whether the product is the original number, using either the exact general form of his factors or the arithmetic form obtained by assuming and substituting values, or both ways, so that he knows what is *meant* by the "factors of a number."

Moreover he is not uncertain regarding the correctness of his work nor is he so likely to leave his factoring in bad shape so common with beginners such as the factors of $ax+bx+ay+by$ are $x(a+b)+y(a+b)$; and $+$ and $-$ signs are less often out of place in factoring $a^3+b^3$, and $a^3-b^3$, etc.

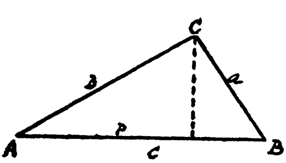
In the solution of verbal problems by use of the equation the value of the test is of course evident. The objection to this plan of testing every example is that fewer problems can be done in a given time, but my experience leads me to say that my first-year classes, after using this method, surpass in rapidity and accuracy of work all former classes, and their pleasure in the work is far greater.

A fourth writes on methods in plane trigonometry which have been found to be very satisfactory.

After a thorough understanding of the functions of an acute angle and the solution of the right-triangle, take up the functions of an obtuse angle including the quadrantal angle 90°. Next, derive the formulas for the solution of the obtuse-angled triangle by methods of plane geometry as follows:

I. *Derivation of the sine law by methods of plane geometry.*—This is given in all texts.

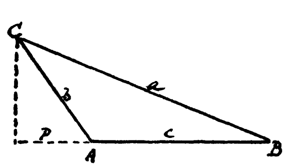
II. *Derivation of the cosine law by methods of plane geometry.*




$a^2=b^2+c^2-2Pc$

$p=b\cos A$

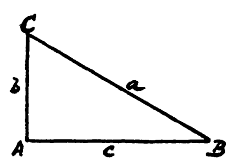
$\therefore a^2=b^2+c^2-2bc\cos A.$




$a^2=b^2+c^2+2Pc$

$p=-b\cos A$

$a^2=b^2+c^2-2bc\,\mathrm{ccs}\,A.$



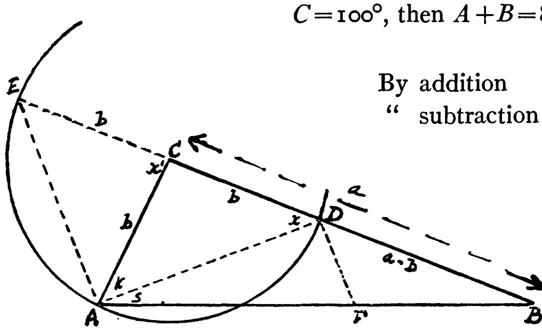

If $A$ is 90° the formulas become

$a^2=b^2+c^2$ (?)

III. *Derivation of tangent law by methods of plane geometry.*

$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{a-b}{a+b}$ or $\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{1}{2}c.$ To solve a $\triangle$ when

two sides and the included $\angle$ are given, $(a, c, b,)$ find first by the above form $\frac{1}{2}(A-B)$. Suppose $\frac{1}{2}(A-B)=10^\circ$, and $C=100^\circ$, then $A+B=80^\circ$, $\frac{1}{2}(A+B)=40^\circ$




$$\frac{1}{2}(A-B)=10^\circ$$

By addition $A = 50^\circ$

" subtraction $B = 30^\circ$

The $\triangle ABC$ can ***now*** be completely solved by aid of sine law constructions. C is the center of arc EAD, $CE=CD=CA$ and $DEA=$ a rt $\angle$ (?) DF is drawn $\parallel$ EA $\therefore$ $\triangle$ DFB and EAB are similar $X'=X+K=2X$ (?) $\therefore X=\frac{1}{2}X'$, $X'=A+B \therefore X=\frac{1}{2}(A+B)$; $s=x-B=\frac{1}{2}(A+B)-B=\frac{1}{2}(A-B)$ (?) EAD and ADF are rt $\triangle$ (DF $\parallel$ EA)

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{\tan s}{\tan x}=\frac{\;\frac{DF}{DA}\;}{\frac{EA}{DA}}=\frac{DF}{EA}=\frac{DB}{EB}=\frac{a-b}{a+b}.$$

$[\frac{1}{2}A+\frac{1}{2}B+\frac{1}{2}C=90^\circ \therefore \tan\frac{1}{2}(A+B)=\cot\frac{1}{2}C]$.

Hence $\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{1}{2}C$ [sim. $\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{1}{2}A$, $\tan\frac{1}{2}(C-A)=\frac{c-a}{c+a}\cot\frac{1}{2}B$].

IV. *Derivation of the formula* $\tan \frac{1}{2} A=\frac{r}{s-a}$ *by methods of Pl. Geometry.*

$[s=\frac{1}{2}(a+b+c)]$

[$s=\frac{1}{2}$ the perimeter]

To prove:

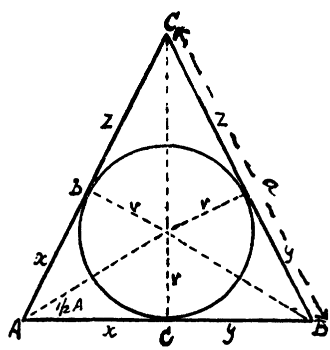
$$\tan\frac{1}{2}A=\frac{r}{s-a} \text{ and } r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$\tan\frac{1}{2}A=\frac{r}{x}$$

$$2x+2y+2z=2s$$

$\therefore x+y+z=s$ $\therefore x=s-(y+z)=s-a$ i. e. $x=s-a$.

Area of $\triangle=F=\frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr=\frac{1}{2}(a+b+c)r=sr$



$$F=\sqrt{s(s-a)(s-b)(s-c)} \quad \text{(by Pl. Geom.)}=sr.$$

$$\therefore r=\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}}=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

Hence $\tan \frac{1}{2}A=\frac{r}{s-a}$, $\tan \frac{1}{2}B=\frac{r}{s-b}$, $\tan \frac{1}{2}C=\frac{r}{s-c}$ where $r=11$

The above proofs are easily' worked by the class and the solution of the oblique triangle "takes hold" at once. Insistence on the graphic method in connection with the above adds to the interest and understanding in no small degree. I might add other methods of presenting other parts of the subject than the orthodox, formal, cut-and-dried ones of the ordinary text, but none has the importance and helpfulness of the above.

A fifth writes:

From persistent practice during my teaching of the past year, I have become thoroughly convinced that the inductive treatment of subject-matter in mathematics is the proper method. By the inductive method I mean the process of leading up to every general statement by the easiest and most gradual concrete examples, allowing the pupil himself to develop the habit of making generalizations. It is remarkable and unfortunate how universal is the procedure in texts and in teaching of stating the general truth usually in very abstract language followed by one or two illustrations. Moreover many texts make the error of attempting to make principles and theorems too general rather than merely sufficiently general for the case in hand. This frequently has the effect of making the principles incomprehensible to the pupil.